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Cycle index series of structures over digraphs

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Abstract

Recently, we introduced (Méndez, Adv. Math. 123 (1996) 243–275.) a generalization of Joyal (Adv. Math. 42 (1981) 1–82) species in order to enumerate structures constructed over arc-labeled directed graphs (species on digraphs). We introduce here the Z-series for species on digraphs, in analogy with Joyal Z-series for ordinary species. We apply the Z-series to obtain classical and new generating functions for unlabeled structures on digraphs. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

Given a class of combinatorial structures M, closed under relabeling isomorphisms (a species, in the sense of Joyal [13]; see also [2]) there are two fundamental generating functions associated to it:

$$M(x)=\sum_{n=1}^{\infty}|M[n]|\frac{x^n}{n!}, \tag{1}$$

$$\tilde{M}(x)=\sum_{n=1}^{\infty}|\tilde{M}[n]|x^n. \tag{2}$$

The first generating function enumerates labeled M-structures, while the second one counts the number of unlabeled M-structures. The coefficient $|M[n]|$ counts the number of M-structures labeled with labels in a set with n elements, and the coefficient $|\tilde{M}[n]|$ the number of objects in $|M[n]|$ after 'erasing' the labels. Usually, the problem of enumerating unlabeled combinatorial objects (or of computing $\tilde{M}(x)$) is much harder than the problem of enumerating labeled combinatorial objects (computing $M(x)$).

Classically, the fundamental tools in the enumeration of unlabeled structures are Burnside lemma and Pólya theory. In the context of Joyal species the generating

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function $\tilde{M}(x)$ is computed by means of the cycle index series $Z_M(x_1,x_2,x_3,\ldots)$ which plays the role of the cycle index polynomial in Pólya theory:

$$\tilde{M}(x) = Z_M(x,x^2,x^3,\ldots).$$

The combinatorial operations on species (sum, product, substitution, derivation, etc.) translates to the analogous algebraic operations on their first generating function (sum, product, substitution, derivation, etc.) and more important, over their Z-series (sum, product, plethysm, partial derivative with respect to x_1, etc.). Many complicated species of structures can be constructed from elementary ones by reiterative applications of combinatorial operations. Some apparently complicated species of structures can be decomposed into elementary ones by the use of combinatorial operations. The homomorphism between combinatorics and algebra provides then a mechanical procedure to obtain complicated Z-series from simpler ones, and hence, a recipe to enumerate the corresponding unlabeled structures.

In [16] we introduced a generalization of the original Joyal species. Such kind of species (species on digraphs) are useful in dealing with the problem of the enumeration of structures which are 'built' over arc-labeled directed graphs (digraphs), in the same way as the ordinary theory of species deals with structures 'built' over sets. In this approach, the adjacency matrix is to a digraph what the cardinal is to a set.

In analogy with the ordinary species, to a species on digraphs D we assign two generating functions in a matrix $\boldsymbol{X}$ of variables

$$D(\boldsymbol{X}) = \sum_A |D[A]|\frac{\boldsymbol{X}^A}{A!}, \tag{3}$$

$$\tilde{D}(\boldsymbol{X}) = \sum_A |\tilde{D}[A]|\boldsymbol{X}^A, \tag{4}$$

where A is a matrix of non-negative integers, $\boldsymbol{X}^A = \prod_{i,j} x_{i,j}^{a_{i,j}}$, $A! = \prod_{i,j} a_{i,j}!$, $|D[A]|$, and $|\tilde{D}[A]|$ are, respectively, the number of labeled and unlabeled D-structures over a digraph with adjacency matrix A.

Many problems in combinatorics have to do with the enumeration of structures with some kind of adjacency pattern. For example the number of words over $\{1,2,\ldots,n\}$ where the adjacent pair (i,j) appears $a_{i,j}$ times (see for example [8]). Other examples of structures with adjacency pattern are the Cartier–Foata flows and rearrangements [3], necklaces (or circular words) [7,18], multisets of necklaces (or multiset permutations), words with adjacency pairs belonging to a family of words over a partition of $\{1,2,\ldots,n\}\times\{1,2,\ldots,n\}$ [5,6,9–12].

From our point of view, all of these examples are unlabeled structures over digraphs. Words with a given adjacency matrix are obtained by unlabeling Eulerian paths of a digraph with the same adjacency matrix (see [8]). The flows and rearrangements are obtained by unlabeling certain kinds of digraphs enriched with linear orders. Since the three previous examples are rigid structures (they have only one automorphism, the identity), their first and second generating functions happen to be the same. This is not true for most of the species on digraphs. For example the necklaces and the multiset permutations are obtained by unlabeling Eulerian cycles and Eulerian permutations over digraphs, respectively. They are not rigid structures and the use of classical Pólya theory does not provide a straightforward and unified method to deal with them.

In Section 2 we give an overview of species on digraphs imbedded into the category of colored species [15,17], and introduce the Z-series for species on digraphs. They depend on an infinite number of matrices of variables, and their second generating function is computed from their Z-series in a way that is very similar to the classical one for ordinary species (see formula (72)). The analogy with Joyal Z-series is striking. For example, the Z-series of the cyclic permutations and permutations in ordinary species are very similar to the Z-series of the Eulerian cycles (see formula (51)) and Eulerian permutations (see formula (52)), respectively. The same may be said about ordinary rooted trees and Eulerian trees, octopuses and their Eulerian analog, etc.

The use of the homomorphism between the combinatorial operations and algebraic operations over the Z-series fulfills the paradigm of enumerating complicated structures from simpler ones. For example, the Eulerian permutations are assemblies of Eulerian cycles. The operation of taking assemblies of structures corresponds to the algebraic operation of taking the exponential of a generating function. The Z-series of the Eulerian permutations is the plethysm of the infinite variables analog of the exponential generating function with the Z-series of the Eulerian cycles (see Theorem (3.2)). Since multiset permutations are obtained by unlabeling Eulerian permutations, and necklaces by unlabeling Eulerian cycles, we establish a logarithmic connection between multiset permutations and necklaces. The problem was originally stated by Goulden and Jackson in [7] in another way. They asked for a combinatorial explanation of the logarithmic connection, that occurs at a generating function level, between words and necklaces. We believe that this combinatorial explanation does not exist, the Eulerian paths can not be expressed as assemblies of Eulerian cycles. The Eulerian paths are rigid structures, the Eulerian cycles are not.

In Section 4 we use some of the Z-series computed in Section 3 to obtain classical and new generating functions for unlabeled structures over digraphs. Of particular interest is the generating function for multiset permutations (formula (74)), which is a matrix-of-variables generalization of Euler's classical generating function for the number of partitions of non-negative integers.

A continuation of this article will be the generalization to this context of Labelle's Γ-series [14] for the enumeration of the rigid (asymmetric) part of a species.

2. Species on digraphs

Definition 2.1. Let I be a finite set. An *I-coloration* (E, c) (or *colored set*) is a finite set E and a function $c: E \to I$ assigning 'colors' in I to E. The cardinal of a colored set (E, c) is defined as the I-tuple $(|c^{-1}(i)|)_{i\in I} \in \mathbb{N}^I$.

The I-colorations and the color-preserving bijections $\tau:(E,c)\to(F,k)$, $k\circ\tau=c$, form a category that we denote by $\mathbb{B}_I$ (see [17]). The direct sum $(E,c_1)+(F,c_2)=(E\uplus F,c_1\uplus c_2)$ of two I colored sets, is the disjoint union of $E\uplus F$ together with the natural common extension $c_1\uplus c_2$ of c_1 and c_2 to $E\uplus F$. Given a set A, the cartesian product $A\overrightarrow{\times}(E,c)$ is the colored set $(A\times E,\hat{c})$, where $\hat{c}(a,e):=c(e)$ for every pair $(a,e)\in A\times E$.

Denote by $[n]$ the set $\{1,2,3,\ldots,n\}$. An arc-labeled digraph G with arc labels set E and vertices in $[n]$ is a colored set $(E,\boldsymbol{v})$:

$$\boldsymbol{v}:E\to[n]\times[n]=[n]^2.$$

The components of the ordered pair $\boldsymbol{v}(e)=(v_1(e),v_2(e))$ are, respectively, the initial and final endpoints of the arc with label $e\in E$. The digraphs with vertex set $[n]$, as colored sets, form a category that we denote by $\mathbb{G}_n:=\mathbb{B}_{[n]^2}$.

Note that the cardinal of a digraph $G=(E,\boldsymbol{v})$, $(|\boldsymbol{v}^{-1}(i,j)|)_{i,j=1}^n$, is its adjacency matrix. The digraph G is called *Eulerian* if for each vertex i the number of arcs pointing to i equals the number of arcs pointing away from i. It is equivalent to the equipotence $|(E,v_1)|=|(E,v_2)|$ of the colored sets (E,v_1) and (E,v_2). Let $G_1=(E,\boldsymbol{v})$ and $G_2=(F,\boldsymbol{w})$ be two digraphs in $\mathbb{G}_n$. Note that the direct sum G_1+G_2 is the directed graph formed by the superposition (disjoint union) of the arcs of G_1 and G_2. The product $G_3=G_1\star G_2$ is the digraph whose arcs are the paths of length two in G_1+G_2, the first arc being in G_1 and the second in G_2. Formally $G_3=(H,\boldsymbol{u})$, where H is the subset of the cartesian product $E\times F$ of pairs (e,f) such that $v_2(e)=w_1(f)$ and $\boldsymbol{u}(e,f)=(v_1(e),w_2(f))$. It is easy to verify that

$$|G_1\star G_2|=|G_1|\,.\,|G_2|,$$

where the symbol . on the right-hand side means matrix product.

An *I-colored J-species* M ((I,J)-species, for short) is a functor

$$M:\mathbb{B}_I\to\mathbb{B}_J.$$

An (I,J)-species M may be identified with a J tuple of $(I,1)$ species $M=(M_j)_{j\in J}$, $M_j:\mathbb{B}_I\to\mathbb{B}$.

A *species on digraphs* is any $([n]^2,J)$-species for some n and some set J. A *matrix species* is any $(I,[n]^2)$-species for some n and some set I.

Note that to each colored set (E,c) (that in particular may be a digraph) a matrix species $\boldsymbol{M}$ associates a digraph $\boldsymbol{M}[E,c]$. The set of arcs form i to j in $\boldsymbol{M}[E,c]$ is denoted by $\boldsymbol{M}_{i,j}[E,c]$. The arcs of $\boldsymbol{M}[E,c]$ are called $\boldsymbol{M}$-structures over (E,c) (Fig. 1).

The generating function of a (I,J)-species is the formal power series with coefficients in $\mathbb{N}^J$:

$$M(x_i)_{i\in I}=\sum_{n\in\mathbb{N}^I}|M[\boldsymbol{n}]|\prod_{i\in I}\frac{x_i^{n_i}}{n_i!},$$

where $|M[\boldsymbol{n}]|$ is the cardinal $|M[E,c]|$, (E,c) being any I-colored set of cardinal $\boldsymbol{n}$. Observe that $M(x_i)_{i\in I}$ can also be thought of as a J-tuple $(M_j(x_i)_{i\in I})_{j\in J}$ of formal power series in $\mathbb{N}[[(x_i)_{i\in I}]]$.

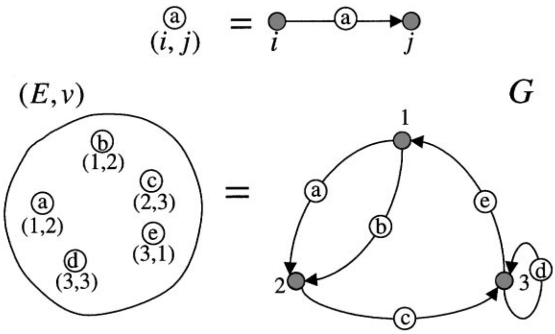

Fig. 1. Digraph and corresponding colored set.

In particular, when $\boldsymbol{M}$ is a matrix species, its generating function is a matrix of formal power series

$$\boldsymbol{M}(x_i)_{i\in I} = \|\boldsymbol{M}_{r,s}(x_i)_{i\in I}\|_{r,s=1}^{n} \in \mathbb{N}^{[n]^2}[[(x_i)_{i\in I}]],$$

where $\mathbb{N}^{[n]^2}$ is the half-ring of $n\times n$ matrices with non-negative integer entries.

If M is a species on digraphs, its generating function is a formal power series over a matrix of variables $M(\boldsymbol{X}) = M(x_{r,s})_{r,s=1}^{n}$.

2.1. Cycle index series

An automorphism $\sigma:(E,c)\to(E,c)$ of an I-coloration is a color-preserving bijection $\sigma:E\to E$, $c\circ\sigma=c$. It means that every cycle of the permutation σ is in $c^{-1}(i)$, for some $i\in I$. Equivalently, every cycle of σ is a 'monochromatic' cycle. Let $a_i^{(k)}$ be the number of cycles in σ of color i and length k. The class of σ is the infinite tuple $\vec{\boldsymbol{a}}=((a_i^{(1)})_{i\in I},(a_i^{(2)})_{i\in I},(a_i^{(3)})_{i\in I},\dots)$. Note that $\vec{\boldsymbol{a}}$ has an infinite number of components, but only a finite number of them are different from zero. We denote by $z_{\vec{\boldsymbol{a}}}$ the number of automorphisms of a permutation of class $\vec{\boldsymbol{a}}$:

$$z_{\vec{\boldsymbol{a}}} = \prod_{i\in I}\prod_{k\in \boldsymbol{P}} k^{a_i^{(k)}} a_i^{(k)}!.$$

Observe that for an (I,J)-species $M:\mathbb{B}_I\to\mathbb{B}_J$, the elements of the J-colored set $M[E,c]$ may be represented as pairs of the form (m,j), where j is the color of the structure m. Since $M[\sigma]$ is a color-preserving bijection, $M[\sigma]m=m'$ implies that the color of m is equal to the color of m'. Then, the set of orbits of $M[E,c]$ under the automorphisms of (E,c) is a colored set whose elements are of the form $(\tilde{m},j)$. The color j is the color of any of the elements of the orbit $\tilde{m}$. We denote this colored set as $\tilde{M}[E,c]$. The generating function of the types or orbits of M-structures is defined to be

$$\tilde{M}(x_i)_{i\in I} = \sum_{\boldsymbol{n}\in\mathbb{N}^I} |\tilde{M}[\boldsymbol{n}]| \prod_{i\in I} x_i^{n_i}.$$

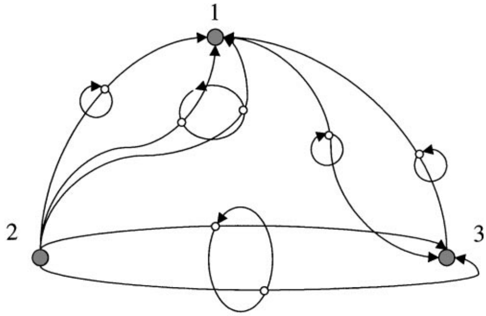

Fig. 2. Digraph with automorphism.

$Z_M[\sigma]$ is defined as the J-colored set of M-structures fixed by σ:

$$Z_M[\sigma] = \{(m, j) \in M[E, c] | M[\sigma]m = m\}.$$

By functoriality, $|Z_M[\sigma]|$ depends only on the class $\vec{\boldsymbol{a}}$ of σ. Denote by $\boldsymbol{x}_k$ the I-tuple of variables $\boldsymbol{x}_k = (x_i^{(k)})_{i \in I}$, $\boldsymbol{x}^{\underline{k}} = (x_i^k)_{i \in I}$, and let $\boldsymbol{x}_k^{\boldsymbol{a}_k} = \prod_{i \in I} (x_i^{(k)})^{a_i^{(k)}}$. $Z_M(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots)$ is the formal power series

$$Z_M(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots) = \sum_{\vec{\boldsymbol{a}}} |Z_M[\vec{\boldsymbol{a}}]| \frac{\boldsymbol{x}_1^{\boldsymbol{a}_1} \boldsymbol{x}_2^{\boldsymbol{a}_2} \ldots}{z_{\vec{\boldsymbol{a}}}}.$$

Proposition 2.1. *The generating function $\tilde{M}(x_i)_{i \in I}$ is obtained by raising the sub-indices to underlined exponents in the series $Z_M(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)$:*

$$\tilde{M}(x_i)_{i \in I} = Z_M(\boldsymbol{x}^{\underline{1}}, \boldsymbol{x}^{\underline{2}}, \boldsymbol{x}^{\underline{3}}, \ldots). \tag{5}$$

Note that in the particular case when $I = [n]^2$, the class of an automorphism $\sigma : G \to G$ of a digraph G is a vector of matrices $\vec{\boldsymbol{A}} = (\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots)$, where $\boldsymbol{A}_k = \| a_{i,j}^{(k)} \|_{i,j=1}^n$, $a_{i,j}^{(k)}$ being the number of k-cycles of σ in $\boldsymbol{v}^{-1}(i, j)$ (see Fig. 2).

If $M : \mathbb{G}_n \to \mathbb{B}_J$ is a species on digraphs, the series Z_M depends on an infinite number of matrices of variables

$$Z_M(\vec{\boldsymbol{X}}) = Z_M(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \ldots) = \sum_{\vec{\boldsymbol{A}}} |Z_M[\vec{\boldsymbol{A}}]| \frac{\boldsymbol{X}_1^{\boldsymbol{A}_1} \boldsymbol{X}_2^{\boldsymbol{A}_2} \ldots}{z_{\vec{\boldsymbol{A}}}}, \tag{6}$$

where

$$\boldsymbol{X}_k = \| x_{r,s}^{(k)} \|_{r,s=1}^n, \quad k = 1, 2, 3 \ldots$$

and

$$z_A = \prod_{k=1}^{\infty} \prod_{i,j=1}^{n} (k)^{a_{i,j}^{(k)}} a_{i,j}^{(k)}!.$$

Example 2.1 (*Constant matrix species on digraphs*). Let $G_0 \in \mathbb{G}_n$ be a fixed digraph with adjacency matrix A. The constant matrix species $\boldsymbol{G_0} : \mathbb{G}_n \to \mathbb{G}_n$ is defined by

$$\boldsymbol{G_0}[G] = \begin{cases} G_0 & \text{if } G = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where 0 is the empty digraph (with an empty set of arcs). The generating function of $\boldsymbol{G_0}$ is the constant matrix A, $\boldsymbol{G_0}(\boldsymbol{X}) = A$. When $G_0 = \mathrm{Id}$, Id being the digraph consisting of a loop on each vertex, the corresponding constant matrix species is denoted as $\boldsymbol{I}$. If G_0 is the digraph with exactly one arc between any ordered pair of vertices, the corresponding constant matrix species will be denoted by $\boldsymbol{J}$.

Example 2.2. Let $\mathfrak{X} : \mathbb{G}_n \to \mathbb{G}_n$ be the singleton matrix species on digraphs:

$$\mathfrak{X}\,[G] = \begin{cases} G & \text{if } G \text{ is a singleton-arc digraph,} \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where 0 is the empty digraph. Denote by $\mathfrak{X}_{i,j} : \mathbb{G}_n \to \mathbb{B}$ the i,j component of $\mathfrak{X}$. We have

$$Z_{\mathfrak{X}}\,(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \ldots) = \boldsymbol{X}_1$$

and

$$Z_{\mathfrak{X}_{i,j}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \ldots) = x_{i,j}^{(1)}.$$

Let S be a subset of $[n]^2$. The restricted singleton species $\mathfrak{X}_S$ is defined by

$$\mathfrak{X}_S[G] = \begin{cases} G & \text{if } G = (\{e\}, \boldsymbol{v}) \text{ is a singleton-arc digraph with } \boldsymbol{v}(e) \in S, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

We have

$$Z_{\mathfrak{X}_S}(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \ldots) = \boldsymbol{X}_{1,S},$$

where for any matrix $\boldsymbol{T} = \|T_{i,j}\|_{i,j=1}^n$, $\boldsymbol{T}_S = \|T_{i,j} \cdot \chi_S(i,j)\|_{i,j=1}^n$, χ_S is the indicator of the set S.

Given a species on digraphs $M : \mathbb{G}_n \to \mathbb{B}_J$, define the matrix species M_{eu} by

$$M_{\mathrm{eu}}[G] = \begin{cases} M[G] & \text{if } G \text{ is Eulerian,} \\ (\emptyset, \emptyset) & \text{otherwise,} \end{cases}$$

where $(\emptyset, \emptyset)$ is the empty colored set.

Denote by $\boldsymbol{sX}$ and by $\boldsymbol{Xs}$, respectively, the product of $\mathrm{diag}(s_1, s_2, \ldots, s_n)$ by $\boldsymbol{X}$ and the product $\boldsymbol{X}$ by $\mathrm{diag}(s_1, s_2, \ldots, s_n)$. The generating function and the Z-series of M and M_{eu} are related in the following way:

$$M_{\mathrm{eu}}(\boldsymbol{X}) = \sum_{\boldsymbol{m} \in \mathbb{N}^n} [\boldsymbol{s}^{\boldsymbol{m}} \boldsymbol{t}^{\boldsymbol{m}}] M(\boldsymbol{s}\,\boldsymbol{X}\boldsymbol{t}), \tag{9}$$

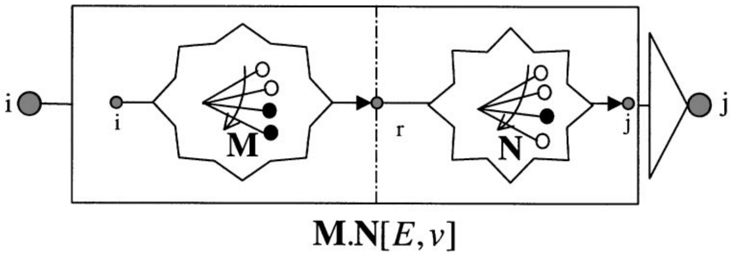

Fig. 3. $\boldsymbol{M}.\boldsymbol{N}$-structure

$$Z_{M_{\mathrm{eu}}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \sum_{\boldsymbol{m} \in \mathbb{N}^n} [\boldsymbol{s}^{\boldsymbol{m}} \boldsymbol{t}^{\boldsymbol{m}}] Z_M(\boldsymbol{s}\boldsymbol{X}_1 \boldsymbol{t}, \boldsymbol{s}^{\underline{2}} \boldsymbol{X}_2 \boldsymbol{t}^{\underline{2}}, \ldots), \tag{10}$$

$$\widetilde{M_{\mathrm{eu}}}(\boldsymbol{X}) = \sum_{\boldsymbol{m} \in \mathbb{N}^n} [\boldsymbol{s}^{\boldsymbol{m}} \boldsymbol{t}^{\boldsymbol{m}}] \tilde{M}(\boldsymbol{s}\, \boldsymbol{X} \boldsymbol{t}), \tag{11}$$

where $[\boldsymbol{s}^{\boldsymbol{m}} \boldsymbol{t}^{\boldsymbol{m}}]$ is the standard coefficient extraction operator (see [6, Section 1.1.2]).

2.2. Combinatorial operations

(1) A family $\{M_s\}_{\in S}$ of (I, J) species is said to be summable if for every colored set (E, c), the set $\{s \in S \mid |M_s[E, c]| \neq \vec{\boldsymbol{0}}\}$ is finite. The sum of the family $\{M_s\}_{s \in S}$ is defined by

$$\left(\sum_{s \in S} M_s\right)[E, c] = \sum_{s \in S} M_s[E, c],$$

where the sum on the right means direct sum of colored sets (see Fig. 3).

(2) Let $\boldsymbol{M}, \boldsymbol{N} : \mathbb{B}_I \to \mathbb{G}_n$ be two matrix species. We define the product $\boldsymbol{M} . \boldsymbol{N}$ by Fig. 3

$$\boldsymbol{M} . \boldsymbol{N}[E, c] = \sum \boldsymbol{M}[E_1, c_1] \star \boldsymbol{N}[E_2, c_2],$$

where the sum on the right (direct sum of products of digraphs) ranges over all ordered decompositions of the colored set (E, c),

$$(E, c) = (E_1, c_1) + (E_2, c_2).$$

(3) Let M be an (I, J) species satisfying $M[\emptyset, \emptyset] = (\emptyset, \emptyset)$. An assembly of M-structures over $(E, c) \in \mathbb{B}_I$ is given by

- A partition π of E and a coloration $k : \pi \to J$. (π, k) is called a *J-colored partition*.
- A structure of $M_{k(B)}[B, c|_B]$ for each block $B \in \pi$.

Let N be a (J, S) species. The substitution $N(M)$ is the (I, S)-species whose (colored)-structures are of the form $((a, n), s)$, where a is an assembly of M structures and n is an element of $N_s[\pi, k]$. In symbols

$$N(M)[E, c] = \sum_{(\pi, k)} \left(\prod_{B \in \pi} M_{k(B)}[B, c|_B]\right) \vec{\times} N[\pi, k],$$

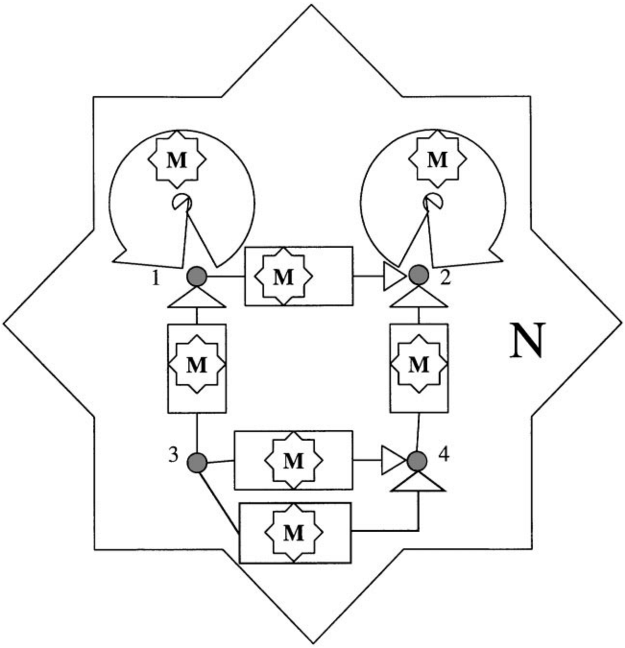

Fig. 4. $N(M)$-structure.

where the symbol $\prod$ on the right is cartesian product, and the sum (direct sum of S-colored sets) ranges over all the J-colored partitions of (E,c).

Observe that a colored partition $(\pi,\boldsymbol{w})$ over a digraph $(E,\boldsymbol{v})$ is another digraph whose arcs are the blocks of π. Then, if $\boldsymbol{M}$ is a matrix species $\boldsymbol{M}:\mathbb{B}_I \rightarrow \mathbb{G}_n$, an assembly of $\boldsymbol{M}$-structures over $(E,c)\in\mathbb{B}_I$ is a 'fat' digraph $(\pi,\boldsymbol{w})$ where each arc $B\in\pi$, $\boldsymbol{w}(B)=(i,j)$, is enriched with an element of $\boldsymbol{M}_{i,j}[B,c|_B]$. The structures of the substitution $N(\boldsymbol{M})$ are 'fat' digraphs whose arcs are $\boldsymbol{M}$-structures (represented in Fig. 4 as $\boldsymbol{M}$-labeled stars) plus an N-structure over the 'fat' digraph (represented as a big star with label N).

(4) Let i be an element of I. For a colored set (E,c), $(E,c)^{+(i)}$ is defined to be the colored set obtained by adding to E a 'ghost' element of color i. The partial derivative and the i-pointing of an (I,J) species M are, respectively, defined by

$$\frac{\partial M}{\partial X_i}[E,c]=M[(E,c)^{+(i)}],$$

$$M^{\bullet(i)}[E,c]=c^{-1}(i)\vec{\times}M[E,c].$$

The cycle index series preserve the classical combinatorial operations on colored species as follows (see [17]):

$$Z_{\sum_{s\in S} M_s}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots)=\sum_{s\in S} Z_{M_s}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots), \tag{12}$$

$$Z_{\boldsymbol{M}\,.\,\boldsymbol{N}}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots) = Z_{\boldsymbol{M}}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots)\,.\,Z_{\boldsymbol{N}}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots), \tag{13}$$

$$\begin{aligned} Z_{N(M)}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots) &= Z_N * Z_M(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots) \\ &= Z_N(Z_M(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots), Z_M(\boldsymbol{x}_2,\boldsymbol{x}_4,\ldots), Z_M(\boldsymbol{x}_3,\boldsymbol{x}_6,\ldots),\ldots), \end{aligned} \tag{14}$$

$$Z_{\partial M/\partial X_i}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots) = \frac{\partial}{\partial x_i^{(1)}} Z_M(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots), \tag{15}$$

$$Z_{M^{\bullet(i)}}(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots) = x_i^{(1)} \frac{\partial}{\partial x_i^{(1)}} Z_M(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots). \tag{16}$$

Definition 2.2. Let $\boldsymbol{M} : \mathbb{G}_n \to \mathbb{G}_n$ be a matrix species on digraphs. The derivative $\frac{d\boldsymbol{M}}{d\mathfrak{X}}$ is the matrix species that assigns to a digraph G the digraph whose arcs from i to j are the elements of the disjoint union

$$\biguplus_{r=1}^{n} \boldsymbol{M}_{r,j}[G + i \xrightarrow{*} r],$$

where $G + i \xrightarrow{*} r$ is the digraph obtained by adding to G an extra 'ghost' arc $i \xrightarrow{*} r$. Clearly $\mathrm{d}\boldsymbol{M}/\mathrm{d}\mathfrak{X}$ is the matrix product of the matrix of operators $\|\partial/\partial\mathfrak{X}_{i,j}\|_{i,j=1}^{n}$ by $\boldsymbol{M}$.

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}\mathfrak{X}} = \left\| \frac{\partial}{\partial\mathfrak{X}_{i,j}} \right\|_{i,j=1}^{n} .\,\boldsymbol{M}.$$

The pointing $\boldsymbol{M}^\bullet$ of $\boldsymbol{M}$ is the matrix species defined by

$$\boldsymbol{M}^\bullet[G] = G{\star}\boldsymbol{M}[G].$$

Proposition 2.2. *Let $\mathfrak{X} \odot \mathrm{d}/\mathrm{d}\mathfrak{X}$ be the operator defined by*

$$\left(\mathfrak{X} \odot \frac{\mathrm{d}}{\mathrm{d}\mathfrak{X}}\right)\boldsymbol{M} = \left\| \mathfrak{X}_{i,j}\frac{\partial}{\partial\mathfrak{X}_{i,j}} \right\|_{i,j=1}^{n} .\,\boldsymbol{M}:$$

Then we have the natural equivalence

$$\boldsymbol{M}^\bullet = \left(\mathfrak{X} \odot \frac{\mathrm{d}}{\mathrm{d}\mathfrak{X}}\right)(\boldsymbol{M}). \tag{17}$$

Proof. Let $G = (E, \boldsymbol{v})$ be any digraph. The set of arcs from i to j in the digraph $(\mathfrak{X} \odot \mathrm{d}/\mathrm{d}\mathfrak{X})(\boldsymbol{M})[G]$ is given by

$$\biguplus_{r=1}^{n} \boldsymbol{M}_{r,j}^{\bullet((i,r))}[G] = \biguplus_{r=1}^{n} \boldsymbol{v}^{-1}(i,r) \times \boldsymbol{M}_{r,j}[G].$$

The disjoint union on the right is the set of arcs from i to j in the digraph $G{\star}\boldsymbol{M}[G]$. $\square$

As an easy consequence of the previous proposition we get the following.

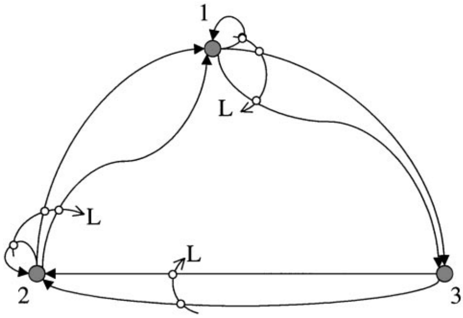

Fig. 5. Rearrangement.

Proposition 2.3. *Let* $\mathrm{d}/\mathrm{d}\boldsymbol{X} = \|\partial/\partial x_{i,j}\|_{i,j=1}^{n}$ *and* $\boldsymbol{X}\odot \mathrm{d}/\mathrm{d}\boldsymbol{X} = \|x_{i,j}\partial/\partial x_{i,j}\|_{i,j=1}^{n}$ *be the operators acting on matrices of generating functions over* $\boldsymbol{X}$ *in an obvious way. Then we have*

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}\mathfrak{X}}(\boldsymbol{X}) = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{X}}\boldsymbol{M}(\boldsymbol{X}), \tag{18}$$

$$Z_{\mathrm{d}\boldsymbol{M}/\mathrm{d}\mathfrak{X}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{X}_1}Z_{\boldsymbol{M}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots), \tag{19}$$

$$\boldsymbol{M}^{\bullet}(\boldsymbol{X}) = \boldsymbol{X}\odot\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{X}}\boldsymbol{M}(\boldsymbol{X}), \tag{20}$$

$$Z_{\boldsymbol{M}^{\bullet}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \boldsymbol{X}_1\odot\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{X}_1}Z_{\boldsymbol{M}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots). \qquad \square \tag{21}$$

3. Examples of cycle index series

Example 3.1 (*Flows and rearrangements*, *Fig.* 5). A *flow* is a digraph $(E, \boldsymbol{v}) \in \mathbb{G}_n$ where each set $v_1^{-1}(r)$, $r\in[n]$ (arcs pointing away from vertex r) is enriched with a linear order l_r. A *rearrangement* is a flow where the subjacent digraph is Eulerian. Denote, respectively, by $\mathfrak{F}$ and $\mathfrak{R}$ the species on digraphs $\mathfrak{F}, \mathfrak{R} : \mathbb{G}_n \to \mathbb{B}$ of flows and rearrangements. Let L be the ordinary species of linear orders. It is easy to verify that

$$\mathfrak{F} = \prod_{i=1}^{n} L\left(\sum_{j=1}^{n}\mathfrak{X}_{i,j}\right).$$

Since $L(x) = 1/(1-x)$ and $Z_L(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots) = 1/(1-x_1)$, we obtain

$$\mathfrak{F}(\boldsymbol{X}) = \prod_{i=1}^{n}\frac{1}{1-\sum_{r=1}^{n}x_{i,r}}$$

and

$$Z_{\mathfrak{F}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \prod_{i=1}^{n} \frac{1}{1 - \sum_{r=1}^{n} x_{i,r}^{(1)}}.$$

Since $\mathfrak{R} = \mathfrak{F}_{\mathrm{eu}}$, by Eqs. (9) and (10)

$$\mathfrak{R}(\boldsymbol{X}) = \sum_{\mathrm{m} \in \mathbb{N}^n} [\boldsymbol{s}^{\boldsymbol{m}}\boldsymbol{t}^{\boldsymbol{m}}] \prod_{i=1}^{n} \frac{1}{1 - \sum_{r=1}^{n} s_i x_{i,r} t_r} = \sum_{\boldsymbol{m} \in \mathbb{N}^n} [\boldsymbol{t}^{\boldsymbol{m}}] \prod_{i=1}^{n} \left(\sum_{r=1}^{n} x_{i,r} t_r \right)^{m_i} \tag{22}$$

and

$$Z_{\mathfrak{R}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \sum_{\boldsymbol{m} \in \mathbb{N}^n} [\boldsymbol{s}^{\boldsymbol{m}}\boldsymbol{t}^{\boldsymbol{m}}] \prod_{i=1}^{n} \frac{1}{1 - \sum_{r=1} s_i x_{i,r}^{(1)} t_r} = \sum_{\boldsymbol{m} \in \mathbb{N}^n} [\boldsymbol{t}^{\boldsymbol{m}}] \prod_{i=1}^{n} \left(\sum_{r=1}^{n} x_{i,r}^{(1)} t_r \right)^{m_i}. \tag{23}$$

The previous examples can be generalized as follows. Let N be an arbitrary ordinary species. Define the N-flows and N-rearrangements by the equations

$$\mathfrak{F}_N = \prod_{i=1}^{n} N \left(\sum_{j=1}^{n} \mathfrak{X}_{i,j} \right)$$

and

$$\mathfrak{R}_N = \mathfrak{F}_{N,\mathrm{eu}}.$$

Clearly, we have

$$\mathfrak{F}_N(\boldsymbol{X}) = \prod_{i=1}^{n} N \left(\sum_{j=1}^{n} x_{i,j} \right), \tag{24}$$

$$Z_{\mathfrak{F}_N}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \prod_{i=1}^{n} Z_N \left(\sum_{j=1}^{n} x_{i,j}^{(1)}, \sum_{j=1}^{n} x_{i,j}^{(2)}, \ldots \right), \tag{25}$$

$$\mathfrak{R}_N(\boldsymbol{X}) = \sum_{\boldsymbol{m} \in \mathbb{N}^n} [\boldsymbol{t}^{\boldsymbol{m}}\boldsymbol{s}^{\boldsymbol{m}}] \prod_{i=1}^{n} N \left(s_i \sum_{j=1}^{n} x_{i,j} t_j \right), \tag{26}$$

$$Z_{\mathfrak{R}_N}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \sum_{\boldsymbol{m} \in \mathbb{N}^n} [\boldsymbol{t}^{\boldsymbol{m}}\boldsymbol{s}^{\boldsymbol{m}}] \prod_{i=1}^{n} Z_N \left(s_i \sum_{j=1}^{n} x_{i,j}^{(1)} t_j, s_i^2 \sum_{j=1}^{n} x_{i,j}^{(2)} t_j^2, \ldots, \right). \tag{27}$$

Example 3.2 (*Eulerian paths*). Consider the matrix species on digraphs

$$\boldsymbol{L} = \boldsymbol{I} + \mathfrak{X} + \mathfrak{X}^2 + \mathfrak{X}^3 + \cdots, \tag{28}$$

$$\boldsymbol{L}_0 = \mathfrak{X} + \mathfrak{X}^2 + \mathfrak{X}^3 + \cdots. \tag{29}$$

By the definition of product of matrix species we easily verify that $\boldsymbol{L}[G]$ is the digraph whose arcs from i to j, $i,j \in [n]$, are the Eulerian paths of G from i to j. $\boldsymbol{L}$ assigns to

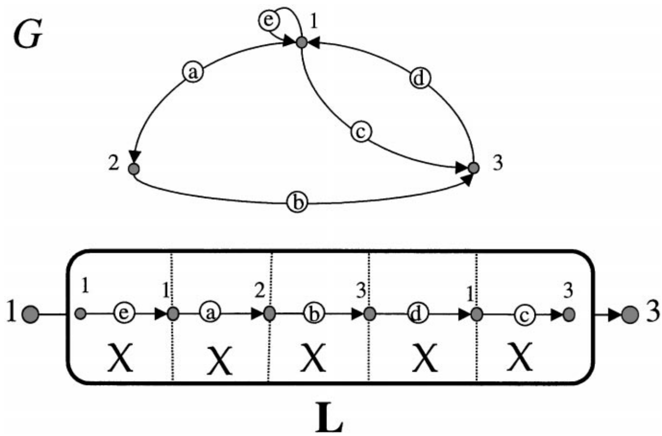

Fig. 6. Structure of $\mathfrak{X}^5 \hookrightarrow \boldsymbol{L}$ over the digraph G.

0 the trivial graph of loops on each vertex. $\boldsymbol{L}_0$ is similar to $\boldsymbol{L}$ except that it sends 0 to 0 (see Fig. 6).

Their generating function and Z-series are

$$\boldsymbol{L}(\boldsymbol{X}) = \frac{\boldsymbol{I}}{\boldsymbol{I}-\boldsymbol{X}}, \tag{30}$$

$$\boldsymbol{L}_0(\boldsymbol{X}) = \frac{\boldsymbol{X}}{\boldsymbol{I}-\boldsymbol{X}}, \tag{31}$$

$$\boldsymbol{Z}_{\boldsymbol{L}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \frac{\boldsymbol{I}}{\boldsymbol{I}-\boldsymbol{X}_1}, \tag{32}$$

$$\boldsymbol{Z}_{\boldsymbol{L}_0}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \frac{\boldsymbol{X}_1}{\boldsymbol{I}-\boldsymbol{X}_1}. \tag{33}$$

Let η be a partition of $[n]^2$. Let $w = B_{i_1} B_{i_2} \ldots B_{i_k}$ be a word in the alphabet η. We say that al Eulerian path $e_1 e_2 \ldots e_k$ over $(E, \boldsymbol{v})$ has η-pattern w, if $\boldsymbol{v}(e_r) \in B_{i_r},\ r = 1, 2, \ldots, k$. By the definition of product of matrix species, the species of Eulerian paths with η-pattern w, $\mathfrak{X}^w$ is given by

$$\mathfrak{X}^w = \mathfrak{X}_{B_{i_1}} \mathfrak{X}_{B_{i_2}} \ldots \mathfrak{X}_{B_{i_k}}.$$

Let $\boldsymbol{L}^{w*}$ be the species of Eulerian paths with η-pattern in $w^* = \{\varepsilon, w, w^2, w^3, \ldots\}$. We have

$$\boldsymbol{L}^{w*} = \boldsymbol{I} + \mathfrak{X}^w + (\mathfrak{X}^w)^2 + (\mathfrak{X}^w)^3 + \dot{s}.$$

Their generating functions and Z-series are

$$\mathfrak{X}^w(\boldsymbol{X}) = \boldsymbol{X}^w, \tag{34}$$

$$\boldsymbol{L}^{w*} = \frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}^{w}}, \tag{35}$$

$$\boldsymbol{Z}_{\mathfrak{X}^{w}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \boldsymbol{X}_1^{w}, \tag{36}$$

$$\boldsymbol{Z}_{\boldsymbol{L}^{w*}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}_1^{w}}. \tag{37}$$

Example 3.3 (*Uniform matrix species, digraphs, and walks*). The uniform matrix species, denoted by $\boldsymbol{U}$, is defined by

$$\boldsymbol{U}[G] = \mathrm{Id}.$$

The generating function and Z-series of $\boldsymbol{U}$ are

$$\begin{aligned}\boldsymbol{U}(\boldsymbol{X}) &= \sum_{A \in \boldsymbol{N}^{[n]^2}} I \prod_{i,j=1}^{n} \frac{x_{i,j}^{a_{i,j}}}{a_{i,j}!} = \mathrm{diag}\left(\exp\left(\sum_{i,j=1}^{n} x_{i,j}\right), \ldots, \exp\left(\sum_{i,j=1}^{n} x_{i,j}\right)\right) \\ &= \exp\left(\sum_{i,j=1}^{n} x_{i,j}\right) I,\end{aligned} \tag{38}$$

$$Z_{\boldsymbol{U}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \exp\left(\sum_{k=1}^{\infty} \sum_{i,j=1}^{n} \frac{x_{i,j}^{(k)}}{k}\right) I, \tag{39}$$

where I is the $n \times n$ identity matrix.

Let $\boldsymbol{G}$ be the matrix species on digraphs defined by

$$\boldsymbol{G}[G] = G, \quad G \in \mathbb{G}_n.$$

We have the natural equivalence

$$\boldsymbol{G} = \mathfrak{X} \,.\, \boldsymbol{U}. \tag{40}$$

From (40) we obtain the generating function and Z-series of $\boldsymbol{G}$,

$$\boldsymbol{G}(\boldsymbol{X}) = \sum_{A \in \mathbb{N}^{[n]^2}} A \prod_{i,j=1}^{n} \frac{x_{i,j}^{a_{i,j}}}{a_{i,j}!} = \left\| x_{i,j} \exp\left(\sum_{r,s=1}^{n} x_{r,s}\right) \right\|_{i,j=1}^{n}, \tag{41}$$

$$Z_{\boldsymbol{G}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \left\| x_{i,j}^{(1)} \exp\left(\sum_{k=1}^{\infty} \sum_{r,s=1}^{n} \frac{x_{r,s}^{(k)}}{k}\right) \right\|_{i,j=1}^{n}. \tag{42}$$

Denote by $\boldsymbol{W}_k$ the matrix species that assigns to any digraph G, the digraph of walks of length k in G. Observe that $\boldsymbol{W}_1 = \boldsymbol{G}$.

Proposition 3.1. *The generating functions and Z-series of $\boldsymbol{W}_k$ satisfy the following equations*:

$$\begin{aligned}\boldsymbol{W}_k(\boldsymbol{X}) &= \left(\boldsymbol{X}\odot \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{X}}\right)^k \operatorname{diag}\left(\exp\left(\sum_{i,j=1}^{n} x_{i,j}\right),\ldots,\exp\left(\sum_{i,j=1}^{n} x_{i,j}\right)\right)\\ &= \left(\boldsymbol{X}\odot \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{X}}\right)^{k-1} \left\| x_{i,j}\exp\left(\sum_{r,s=1}^{n} x_{r,s}\right)\right\|_{i,j=1}^{n},\end{aligned} \tag{43}$$

$$\begin{aligned}Z_{\boldsymbol{W}_k}(\boldsymbol{X}_1,\boldsymbol{X}_2,\ldots) &= \left(\boldsymbol{X}_1\odot \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{X}_1}\right)^k \left(\exp\left(\sum_{k=1}^{\infty}\sum_{i,j=1}^{n} \frac{x_{i,j}^{(k)}}{k}\right) I\right)\\ &= \left(\boldsymbol{X}_1\odot \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{X}_1}\right)^{k-1} \left\| x_{i,j}^{(1)}\exp\left(\sum_{k=1}^{\infty}\sum_{r,s=1}^{n} \frac{x_{r,s}^{(k)}}{k}\right)\right\|_{i,j=1}^{n}.\end{aligned} \tag{44}$$

Proof. By the definition of the operator $\mathfrak{X}\odot\mathfrak{D}$, the matrix species

$$((\ldots(\overbrace{\boldsymbol{U}^{\bullet})^{\bullet})^{\bullet}\ldots)^{\bullet}}^{k} = (\mathfrak{X}\odot\mathfrak{D})^k\boldsymbol{U} = (\mathfrak{X}\odot\mathfrak{D})^{k-1}\boldsymbol{G}$$

assigns to G the graph $G^{\star k} = \overbrace{G\star G\star\cdots\star G}^{k}$. From the definition of the star product of digraphs, $G^{\star k}$ is the digraph of walks of length k over G. This yields to the isomorphism

$$\boldsymbol{W}_k = (\mathfrak{X}\odot\mathfrak{D})^k\boldsymbol{U} = (\mathfrak{X}\odot\mathfrak{D})^{k-1}\boldsymbol{G}. \tag{45}$$

Taking generating functions and Z-series in (45) we obtain the result. □

The matrix analog of the exponential polynomials $\Phi_k(\boldsymbol{X})$ is defined by the equation

$$\Phi_k(\boldsymbol{X}) = \boldsymbol{U}(\boldsymbol{X})^{-1}\left[\left(\boldsymbol{X}\odot\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{X}}\right)^k \boldsymbol{U}(\boldsymbol{X})\right],$$

Properties of $\Phi_k(\boldsymbol{X})$ and applications to enumeration of surjective walks will be studied in a forthcoming paper.

Example 3.4 (*Eulerian cycles and permutations*). An *Eulerian permutation* ς over a digraph $G=(E,\boldsymbol{v})$ is a color-preserving permutation $\varsigma:(E,v_2)\to(E,v_1)$. An *Eulerian cycle* ξ is an Eulerian cyclic permutation $\xi:(E,v_2)\to(E,v_1)$. Denote respectively, by $\mathscr{C}$ and $\mathscr{S}$ the species on digraphs $\mathscr{C},\mathscr{S}:\mathbb{G}_n\to\mathbb{B}$ of Eulerian cycles and Eulerian permutations. Since any Eulerian permutation is an assembly of Eulerian cycles, we have the combinatorial identity

$$\mathscr{S} = U(\mathscr{C}), \tag{46}$$

where U is the ordinary uniform species. The generating function of the Eulerian cycles of length k is the trace of the matrix $\boldsymbol{X}^k/k$. By the Jacobi identity,

$$\mathscr{C}(\boldsymbol{X}) = \sum_{k=1}^{\infty} \operatorname{tr} \frac{\boldsymbol{X}^k}{k} = \operatorname{tr} \ln\left(\frac{\boldsymbol{I}}{\boldsymbol{I}-\boldsymbol{X}}\right) = \ln\left|\frac{\boldsymbol{I}}{\boldsymbol{I}-\boldsymbol{X}}\right|. \tag{47}$$

By Eq. (46) we obtain

$$\mathscr{S}(\boldsymbol{X}) = \exp\left(\ln\left|\frac{\boldsymbol{I}}{\boldsymbol{I}-\boldsymbol{X}}\right|\right) = \left|\frac{\boldsymbol{I}}{\boldsymbol{I}-\boldsymbol{X}}\right|. \tag{48}$$

Eulerian permutations have been introduced in the literature by Strehl [19] under the name of *compatible permutations* (see [19, Section 4.7]). It is also implicit in Gessel's proof of the multivariate Lagrange inversion formula [4]. Formula (48) is proved directly in [19], without using the Jacobi identity (see also [2, Section 3.2]).

The species $\mathfrak{R}$ and $\mathscr{S}$ are generalizations to this context of the ordinary species of linear orders L and permutations S, respectively. Similarly to L and S, $\mathfrak{R}$ and $\mathscr{S}$ are clearly equipotent:

$$|\mathfrak{R}[G]| = |\mathscr{S}[G]| = \begin{cases} \prod_{i=1}^{n} |v_1^{-1}(i)|! & \text{if } G=(E,\boldsymbol{v}) \text{ is Eulerian,} \\ 0 & \text{otherwise.} \end{cases}$$

Since this equipotence is equivalent to the identity $\mathfrak{R}(\boldsymbol{X}) = \mathscr{S}(\boldsymbol{X})$, from Equations (22) and (48) the MacMahon master theorem can be re-stated as follows.

Theorem 3.1. *The species of rearrangements* $\mathfrak{R}$ *and of Eulerian permutations* $\mathscr{S}$ *are equipotent*:

$$\mathfrak{R}(\boldsymbol{X}) = \sum_{\boldsymbol{m}\in\mathbb{N}^n} [\boldsymbol{t}^{\boldsymbol{m}}] \prod_{i=1}^{n} \left(\sum_{r=1}^{n} x_{i,r} t_r\right)^{m_i} = \left|\frac{\boldsymbol{I}}{\boldsymbol{I}-\boldsymbol{X}}\right| = \mathscr{S}(\boldsymbol{X}). \tag{49}$$

By the MacMahon master theorem and Eq. (23)

$$Z_{\mathfrak{R}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \left|\frac{\boldsymbol{I}}{\boldsymbol{I}-\boldsymbol{X}_1}\right|. \tag{50}$$

However, $\mathfrak{R}$ and $\mathscr{S}$ are not isomorphic. Their Z-series are not the same.

Theorem 3.2. *The cycle index series of* $\mathscr{C}$ *and* $\mathscr{S}$ *are*, *respectively*,

$$Z_{\mathscr{C}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln\left|\frac{\boldsymbol{I}}{\boldsymbol{I}-\boldsymbol{X}_k}\right|, \tag{51}$$

$$Z_{\mathscr{S}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \prod_{k=1}^{\infty} \left|\frac{\boldsymbol{I}}{\boldsymbol{I}-\boldsymbol{X}_k}\right|. \tag{52}$$

Proof. To prove the first identity we have to compute

$$|Z_{\mathscr{C}}[\vec{\boldsymbol{A}}]| = |\{\xi \in \mathscr{C}[E, \boldsymbol{v}]\colon \sigma\xi\sigma^{-1} = \xi\}| = |\{\xi \in \mathscr{C}[E, \boldsymbol{v}]\colon \xi\sigma\xi^{-1} = \sigma\}|,$$

where σ is any automorphism of the digraph $G = (E, \boldsymbol{v})$ of class $\vec{\boldsymbol{A}}$. Note that if an Eulerian cycle ξ commutes with σ, it induces a cyclic permutation $\hat{\xi}: \pi \to \pi$, $\hat{\xi}(B) = \xi(B)$, where π is the partition of E induced by the cycles of σ. Then, all the cycles of σ are of the same length and we have that $|Z_{\mathscr{C}}[\vec{\boldsymbol{A}}]| \neq 0$ only when $\vec{\boldsymbol{A}}$ is of the form $\vec{\boldsymbol{A}} = (\boldsymbol{O}, \boldsymbol{O}, \ldots, \boldsymbol{A}_k, \boldsymbol{O}, \ldots)$ for some $k = 1, 2, 3, \ldots$. Now, define the digraph obtained by collapsing the arcs of G that are in the same cycle of σ,

$$\hat{G} = (\pi, \hat{\boldsymbol{v}}), \qquad \hat{\boldsymbol{v}}(B) = \boldsymbol{v}(b), \quad B \in \pi,$$

where b is any element of B.

Observe that

- The cardinal (incidence matrix) of $\hat{G}$ is $\boldsymbol{A}_k$ and that $\hat{\xi}$ defines an Eulerian cycle on $\hat{G}$.
- If $m = |\pi|$ is the length of the Eulerian cycle $\hat{\xi} = (B, \xi(B), \xi^2(B), \ldots, \xi^{m-1}(B))$, B being some block of π, ξ defines $m-1$ bijections $\xi_i : \xi^{i-1}(B) \to \xi^i(B)$, $\xi_i = \xi|_{\xi^{i-1}(B)}$, $i = 1, 2, \ldots, m-1$. Each ξ_i is an isomorphism between the cyclic permutations (of length k) $(\xi^{i-1}(B), \sigma|_{\xi^{i-1}(B)})$ and $(\xi^i(B), \sigma|_{\xi^i(B)})$.
- ξ defines a cyclic permutation $\xi_m : B \to B$, $\xi_m = \xi^m|_B$, commuting with $\sigma|_B$. Since $\sigma|_B$ is cyclic, ξ_m has to be of the form $(\sigma|_B)^r$, where r is some positive integer coprime with k.

Conversely, the reader may verify that from an Eulerian cycle $(B_0, B_1, \ldots, B_{m-1})$ of $\hat{G}$, a sequence of bijections $\xi_i : B_{i-1} \to B_i$, $i = 1, 2, \ldots, m-1$, and $\xi_m : B_0 \to B_0$ as above, we can uniquely recover an Eulerian permutation ξ of G commuting with σ. There are k ways of choosing each ξ_i, $i = 1, 2, \ldots, m-1$, and $\phi(k)$ ways of choosing ξ_m. Since the incidence matrix of $\hat{G}$ is $\boldsymbol{A}_k$, there are $|\mathscr{C}[\boldsymbol{A}_k]|$ ways of choosing the Eulerian cycle $(B_0, B_1, \ldots, B_{m-1})$. Then,

$$|Z_{\mathscr{C}}[\vec{\boldsymbol{A}}]| = \begin{cases} \phi(k)k^{m-1}|\mathscr{C}[\boldsymbol{A}_k]| & \text{if } \vec{\boldsymbol{A}} = (\boldsymbol{O}, \ldots, \boldsymbol{A}_k, \boldsymbol{O}, \ldots) \text{ for some } k, \\ 0 & \text{otherwise,} \end{cases}$$

where $m = \sum_{i,j=1}^{n} a_{i,j}^{(k)}$.

We obtain

$$\begin{aligned} Z_{\mathscr{C}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) &= \sum_{k,m=1}^{\infty} \sum_{\boldsymbol{A}_k, \sum_{i,j} a_{i,j}^{(k)} = m} \phi(k)k^{m-1}|\mathscr{C}[\boldsymbol{A}_k]| \frac{\boldsymbol{X}_k^{\boldsymbol{A}_k}}{\prod_{i,j=1}^{n} (k)^{a_{i,j}^{(k)}} a_{i,j}^{(k)}!} \\ &= \sum_{k,m=1}^{\infty} \sum_{\boldsymbol{A}_k, \sum_{i,j} a_{i,j}^{(k)} = m} \phi(k)k^{m-1}|\mathscr{C}[\boldsymbol{A}_k]| \frac{\boldsymbol{X}_k^{\boldsymbol{A}_k}}{k^m \prod_{i,j=1}^{n} a_{i,j}^{(k)}!} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \sum_{\boldsymbol{A}_k} |\mathscr{C}[\boldsymbol{A}_k]| \frac{\boldsymbol{X}_k^{\boldsymbol{A}_k}}{\prod_{i,j=1}^{n} a_{i,j}^{(k)}!} \\
&= \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln \left| \frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}_k} \right|. \qquad (53)
\end{aligned}
$$

The Z-series of the ordinary uniform species U is well known to be

$$
Z_U(x_1, x_2, x_3, \ldots) = \exp\left(\sum_{i=1}^{\infty} \frac{x_i}{i} \right).
$$

By identity (46)

$$
\begin{aligned}
Z_{\mathscr{S}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) &= Z_U * \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln \left| \frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}_k} \right| \\
&= \exp\left(\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{\phi(k)}{k.i} \ln \left| \frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}_{i.k}} \right| \right) \\
&= \exp\left(\sum_{m=1}^{\infty} \ln \left| \frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}_m} \right| \right) \\
&= \prod_{m=1}^{\infty} \left| \frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}_m} \right|. \qquad (54)
\end{aligned}
$$

Example 3.5 (*Eulerian cycles and permutations with η-patterns*). Let A be an alphabet. Two words w, w' over A are said to be *conjugate* if there exist a pair of non-empty words u and v such that $w = uv$ and $w' = vu$. The conjugacy is an equivalence relation. An equivalence class of words under conjugacy is called a *necklace* (see [18]). We denote by (w) the necklace corresponding to the word w. A word w is called *periodic* if $w = v^j$ for some word v and some $j > 1$. An equivalence class of aperiodic words is called a *primitive necklace*.

Let $\xi : (E, v_2) \to (E, v_1)$ be an Eulerian cycle of length k over $G = (E, \boldsymbol{v})$. For a partition η of $[n]^2$, the η-pattern of ξ is the necklace over η,

$$
([\boldsymbol{v}(e)], [\boldsymbol{v}(\xi(e))], \ldots, [\boldsymbol{v}(\xi^{k-1}(e))]),
$$

where $[(i,j)]$ is the unique block of η such that $(i,j) \in [(i,j)]$. The species of Eulerian cycles with η-pattern (w) is denoted by $\mathscr{C}^{(w)}$. The species of Eulerian cycles with η-pattern in $(w^*) = \{\varepsilon, (w), (w^2), (w^3), \ldots\}$ is denoted by $\mathscr{C}^{(w^*)}$. An Eulerian permutation is said to have η-pattern in (w^*) if each cycle in it has η-pattern in (w^*). The species of Eulerian permutations with η-pattern in (w^*) is denoted by $\mathscr{S}^{(w^*)}$.

The following facts are easy to verify.

Proposition 3.2.

(1) *If the* η*-pattern of* ξ *is a primitive necklace, then* ξ *is asymmetric* (*the only automorphism of* ξ *is the identity*).
(2) *If the* η *pattern of* ξ *is a primitive necklace and* $\eta' \leqslant \eta$ *in the refinement order, then the* η'*-pattern of* ξ *is also primitive.*
(3) *The* $\hat{0}$*-pattern of* ξ *is primitive if and only if* ξ *is asymmetric.*

For a primitive word w, $\mathscr{C}^{(w)}$ is isomorphic to the species $\operatorname{tr} \mathfrak{X}^{\,w}$ of circular Eulerian paths of η-pattern w. We also have the following isomorphisms:

$$\mathscr{C}^{(w^*)} = \mathscr{C}(\mathfrak{X}^{\,w}),$$

$$\mathscr{S}^{(w^*)} = \mathscr{S}(\mathfrak{X}^{\,w}),$$

from which we obtain the generating functions

$$\mathscr{C}^{(w^*)}(\boldsymbol{X}) = \ln \left| \frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}^w} \right|, \tag{55}$$

$$\mathscr{S}^{(w^*)}(\boldsymbol{X}) = \left| \frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}^w} \right|, \tag{56}$$

$$Z_{\mathscr{C}^{(w^*)}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln \left| \frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}_k^w} \right|, \tag{57}$$

$$Z_{\mathscr{S}^{(w^*)}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \prod_{k=1}^{\infty} \left| \frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}_k^w} \right|. \tag{58}$$

Example 3.6 (*Eulerian paths, cycles and permutations over enriched digraphs*). Let $\|M_{i,j}(\mathfrak{X}_{i,j})\|_{i,j=1}^n = \boldsymbol{M} : \mathbb{G}_n \to \mathbb{G}_n$ be a matrix species where in each entry $M_{i,j}$ is an ordinary species satisfying $M_{i,j}[\emptyset] = \emptyset$, $i,j = 1,2,\ldots,n$. If $(E, \boldsymbol{v})$ is a parallel-arc digraph ($|\boldsymbol{v}(E)| = 1$) $\boldsymbol{M}$ assigns to $(E, \boldsymbol{v})$ the parallel-arc digraph whose arcs from i to j are the elements of $M_{i,j}[E]$, (i,j) being the common endpoints of the arcs in E:

$$\boldsymbol{M}[E, \boldsymbol{v}] = (\boldsymbol{M}_{i,j}[E], \boldsymbol{k}), \qquad (i,j) = \boldsymbol{v}(E) = \boldsymbol{k}(M_{i,j}[E]).$$

$\boldsymbol{M}$ assigns to the empty digraph and to any digraph with a couple of non-parallel arcs, the empty digraph

$$\boldsymbol{M}[E, \boldsymbol{v}] = 0 \quad \text{if } |\boldsymbol{v}(E)| \neq 1.$$

A partition π of the arcs of a digraph $G = (E, \boldsymbol{v})$ is said to be incidence-preserving if any two elements in the same block of π have the same initial an final endpoints. The partition π induces a digraph $\hat{G}$ by collapsing the arcs in the same block of π, $\hat{G} = (\pi, \hat{\boldsymbol{v}})$, where $\hat{\boldsymbol{v}}(B) = \boldsymbol{v}(B)$.

An $\boldsymbol{M}$-enriched digraph is the digraph induced by an incidence-preserving partition plus an element of $M_{i,j}[B]$, $(i,j) = \boldsymbol{v}(B)$, on each block B of π (see Fig. 7). In other

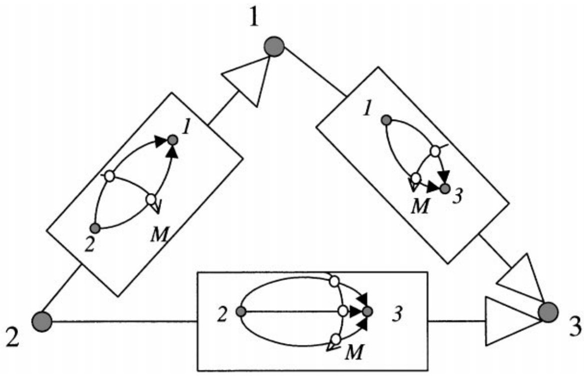

Fig. 7. M-enriched digraph.

words, the arcs of an $\boldsymbol{M}$-enriched digraph from i to j are the elements of an assembly of $M_{i,j}$-structures over the set $\boldsymbol{v}^{-1}(i,j)$.

The reader may verify that an assembly of $\boldsymbol{M}$-structures is an $\boldsymbol{M}$-enriched digraph. The structures of the substitution $N(\boldsymbol{M})$ of $\boldsymbol{M}$ in any matrix species $N : \mathbb{G}_n \to \mathbb{B}_J$ are N-structures over $\boldsymbol{M}$-enriched digraphs.

For example $\boldsymbol{L}(\boldsymbol{M})$ is the matrix species of Eulerian paths over $\boldsymbol{M}$-enriched digraphs. $\mathscr{C}(\boldsymbol{M})$ and $\mathscr{S}(\boldsymbol{M})$ are, respectively, the species of Eulerian cycles and permutations over $\boldsymbol{M}$-enriched digraphs.

We obtain

$$Z_{\boldsymbol{L}(\boldsymbol{M})}(\boldsymbol{X}_1,\boldsymbol{X}_2,\ldots) = \frac{\boldsymbol{I}}{\|\delta_{i,j} - Z_{M_{i,j}}(x_{i,j}^{(1)},x_{i,j}^{(2)},\ldots)\|_{i,j=1}^{n}}, \tag{59}$$

$$Z_{\mathscr{C}(\boldsymbol{M})}(\boldsymbol{X}_1,\boldsymbol{X}_2,\ldots) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln(|\delta_{i,j} - Z_{M_{i,j}}(x_{i,j}^{(k)},x_{i,j}^{(2k)},x_{i,j}^{(3k)},\ldots)|_{i,j=1}^{n})^{-1}, \tag{60}$$

$$Z_{\mathscr{S}(\boldsymbol{M})}(\boldsymbol{X}_1,\boldsymbol{X}_2,\ldots) = \prod_{k=1}^{\infty} (|\delta_{i,j} - Z_{M_{i,j}}(x_{i,j}^{(k)},x_{i,j}^{(2k)},x_{i,j}^{(3k)},\ldots)|_{i,j=1}^{n})^{-1}. \tag{61}$$

Example 3.7 (*Bergeron's Octopuses and assemblies of octopuses, Bergeron* [1]). An octopus is an Eulerian cycle of Eulerian paths. Equivalently, the matrix species of octopuses is the substitution of $\boldsymbol{L}_0$ in $\mathscr{C}$:

$$\mathrm{Oc} = \mathscr{C}(\boldsymbol{L}_0).$$

Let $U(\mathrm{Oc}) = \mathscr{S}(\boldsymbol{L}_0)$ be the species of assemblies of octopuses. From the above identities we obtain (see Fig. 8)

$$\mathrm{Oc}(\boldsymbol{X}) = \ln\left|\frac{\boldsymbol{I}-\boldsymbol{X}}{\boldsymbol{I}-2\boldsymbol{X}}\right|, \tag{62}$$

$$U(\mathrm{Oc})(\boldsymbol{X}) = \left|\frac{\boldsymbol{I}-\boldsymbol{X}}{\boldsymbol{I}-2\boldsymbol{X}}\right|, \tag{63}$$

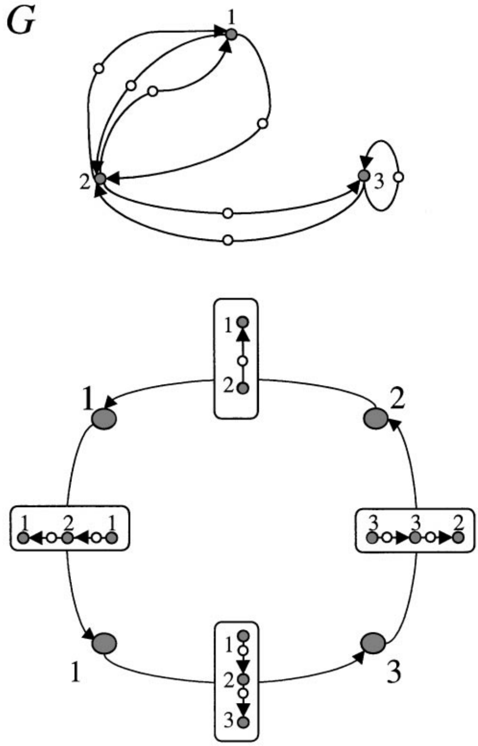

Fig. 8. Octopus, $\mathrm{Oc} = \mathscr{C}(\boldsymbol{L}_0)$, over the digraph G.

$$Z_{\mathrm{Oc}}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln \left| \frac{\boldsymbol{I} - \boldsymbol{X}_k}{\boldsymbol{I} - 2\boldsymbol{X}_k} \right|, \tag{64}$$

$$Z_{U(\mathrm{Oc})}(\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots) = \prod_{k=1}^{\infty} \left| \frac{\boldsymbol{I} - \boldsymbol{X}_k}{\boldsymbol{I} - 2\boldsymbol{X}_k} \right|. \tag{65}$$

Example 3.8 (*Eulerian endofunctions, trees and vertebrates*). An Eulerian endofunction over a digrap G is a color-preserving endofunction

$$\eta : (E, v_2) \to (E, v_1).$$

An (i, j)-Eulerian rooted tree is an acyclic color-preserving function (see Fig. 9)

$$\tau : (E - \{e\}, v_2|_{E-\{e\}}) \to (E, v_1),$$

where the arc e (the root) goes from vertex i to vertex j. We denote by $\boldsymbol{A} : \mathbb{G}_n \to \mathbb{G}_n$ the matrix species of Eulerian trees. $\boldsymbol{A}[G]$ is the digraph whose arcs from i to j are the (i, j)-Eulerian trees over G. The (i, j)-Eulerian trees satisfy the following combinatorial

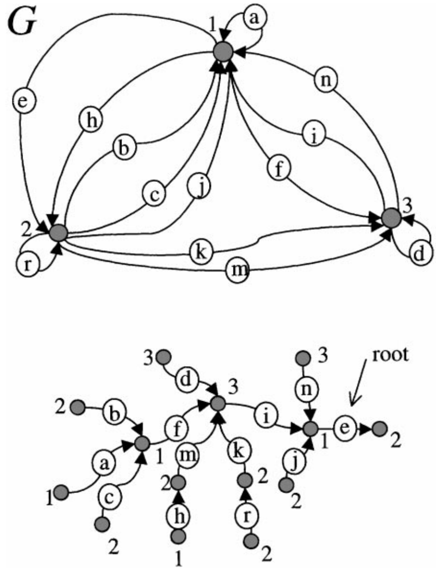

Fig. 9. Eulerian rooted tree over G.

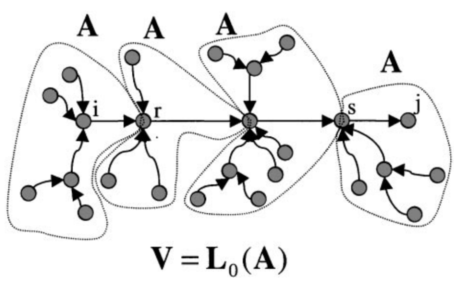

Fig. 10. Vertebrate.

equation:

$$\boldsymbol{A}_{i,j} = \mathfrak{X}_{i,j} \,.\, U(\boldsymbol{A}_{1,i} + \boldsymbol{A}_{2,i} + \cdots + \boldsymbol{A}_{n,i}),$$

where U is the ordinary uniform species.

The species on digraphs $\mathrm{End} : \mathbb{G}_n \to \mathbb{B}$ of Eulerian endofunctions is isomorphic to the substitution of $\boldsymbol{A}$ into $\mathscr{S}$:

$$\mathrm{End} = \mathscr{S}(\boldsymbol{A}).$$

The matrix species of Eulerian vertebrates $\boldsymbol{V} : \mathbb{G}_n \to \mathbb{G}_n$ is defined as the substitution of $\boldsymbol{A}$ into $\boldsymbol{L}_0$ (see Fig. 10):

$$\boldsymbol{V} = \boldsymbol{L}_0(\boldsymbol{A}) = \boldsymbol{A} + \boldsymbol{A}^2 + \boldsymbol{A}^3 + \cdots \,.$$

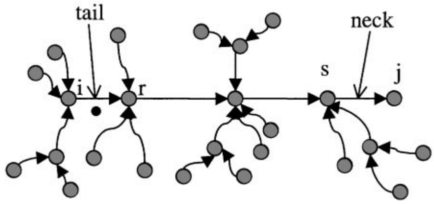

Fig. 11. The identity $V_{i,j}=\sum_{r,s=1}^{n} A_{s,j}^{\bullet(i,r)}$.

It is not difficult to verify that (see Fig. 11)

$$V_{i,j}=\sum_{r,s} A_{s,j}^{\bullet(i,r)}=\sum_{r,s} \mathfrak{X}_{i,r}\frac{\partial}{\partial \mathfrak{X}_{i,r}} A_{s,j}=\sum_{r=1}^{n} \frac{\partial}{\partial \mathfrak{X}_{i,r}}\sum_{s=1}^{n} A_{s,j}.$$

Form the last identity we get.

Proposition 3.3. *We have the combinatorial identity*

$$V=(J\,.\,A)^{\bullet},$$

where J *is the constant matrix species defined in Example* 2.1.

From the above proposition and equations we obtain

$$A_{i,j}(X)=x_{i,j}\exp\left(\sum_{r=1}^{n} A_{r,i}(X)\right), \tag{66}$$

$$\mathrm{End}(X)=\left|\frac{I}{I-A(X)}\right|, \tag{67}$$

$$V(X)=\frac{A(X)}{I-A(X)}=X\odot\frac{\mathrm{d}}{\mathrm{d}X}(J\,.\,A(X)), \tag{68}$$

$$Z_{A_{i,j}}(X_1,X_2,\ldots)=x_{i,j}^{(1)}\exp\left(\sum_{k=1}^{\infty}\sum_{r=1}^{n}\frac{Z_{A_{r,i}}(X_k,X_{2k},\ldots)}{k}\right), \tag{69}$$

$$Z_{\mathrm{End}}(X_1,X_2,\ldots)=\prod_{k=1}^{\infty}\left|\frac{I}{I-Z_A(X_k,X_{2k},\ldots)}\right|, \tag{70}$$

$$Z_V(X_1,X_2,\ldots)=\frac{Z_A(X_1,X_2,\ldots)}{I-Z_A(X_1,X_2,\ldots)}=X_1\odot\frac{\mathrm{d}}{\mathrm{d}X_1}(J\,.\,Z_A(X_1,X_2,\ldots)). \tag{71}$$

4. Enumeration of unlabeled structures

Let D be a species on digraphs. The generating function $\tilde{D}(X)$ counts the number of types of D-structures under the action of the automorphisms in the groupoid of digraphs. It means that the coefficient of $\boldsymbol{X}^A$ of $\tilde{D}(\boldsymbol{X})$ is the number of combinatorial objects obtained by 'erasing' the labels of the structures of $D[G]$, G being any digraph with adjacency matrix A. Such types of structures are said to have adjacency pattern A. By (2.1) the generating function of the unlabeled D-structures is given by

$$\tilde{D}(\boldsymbol{X}) = Z_D(\boldsymbol{X}^{\underline{1}}, \boldsymbol{X}^{\underline{2}}, \ldots). \tag{72}$$

In this section we use Eq. (72) and some of the Z-series obtained in the previous section to compute some classical and new generating functions of classes of unlabeled structures over digraphs. This is only a glimpse over the wide range of possible application of the Z-series of structures over digraphs. We leave for a forthcoming paper the applications to the enumeration of unlabeled trees, vertebrates, endofunctions, etc.

4.1. Generalized rearrangements

We can easily verify that the unlabeled structure corresponding to an element of $\mathfrak{R}$ is a rearrangement (w, w') in the sense of Cartier–Foata [3]. For example, the rearrangement corresponding to the structure represented in Fig. 5 is the following:

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 2 & 1 & 1 & 2 & 2 \end{pmatrix}.$$

By Eq. (50) the generating function is

$$\tilde{\mathfrak{R}}(\boldsymbol{X}) = \left| \frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}} \right| = \mathfrak{R}(\boldsymbol{X}).$$

By Eq. (27) the generating function of the unlabeled N-rearrangements is

$$\tilde{\mathfrak{R}}_N(\boldsymbol{X}) = \sum_{\boldsymbol{m} \in \mathbb{N}} [\boldsymbol{t}^{\boldsymbol{m}} \boldsymbol{s}^{\boldsymbol{m}}] \left[\prod_{i=1}^{n} Z_N \left(s_i \sum_{j=1}^{n} x_{i,j} t_j, s_i^2 \sum_{j=1}^{n} x_{i,j}^2 t_j^2, \ldots \right) \right].$$

For example, for the species S of permutations, the unlabeled structures enumerated by $\tilde{\mathfrak{R}}_S(\boldsymbol{X})$ are objects of the form

$$(\sigma_1)_1 (\sigma_2)_2 \ldots (\sigma_n)_n,$$

where each σ_i is a multiset permutation (a multiset of necklaces) over the set of symbols $[n]$. For each $i \in [n]$ the length $|\sigma_i|$ equals the number of appearances of the symbol i on the whole multiset permutation $\sigma_1\sigma_2\sigma_3 \ldots \sigma_n$. An example of such kind of structures is $((13)(322))_1((331)(11))_2((312)(22))_3$, that can be represented in Cartier–Foata notation as follows:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 \\ (1 & 3) & (3 & 2 & 2) & (3 & 3 & 1) & (1 & 1) & (3 & 1 & 2) & (2 & 2) \end{pmatrix}.$$

Since $Z_S(x_1, x_2, x_3, \ldots) = \prod_{k=1}^{\infty} 1/(1 - x_k)$ we have

$$\tilde{\mathfrak{R}}_S(\boldsymbol{X}) = \sum_{\boldsymbol{m} \in \mathbb{N}^n} [\boldsymbol{t}^{\boldsymbol{m}} \boldsymbol{s}^{\boldsymbol{m}}] \left[\prod_{k=1}^{\infty} \prod_{i=1}^{n} \frac{1}{1 - \sum_{j=1}^{n} s_i^k x_{i,j}^k t_j^k} \right].$$

4.2. Words and words with a given pattern

The unlabeled structure corresponding to an Eulerian path is a word over the alphabet $[n]$. Let $w = C_{i_1}C_{i_2}\cdots C_{i_{l-1}}$ be a word over a the elements of a partition η of $[n]^2$. A word $k_1k_2\ldots k_l$ over $[n]$ is said to have η-pattern w if $(k_r, k_{r+1}) \in C_{i_r}$, $r = 1, 2, \ldots, l-1$. By Eqs. (36) and (37), the generating function of the words with η-pattern w and with η-pattern in w^* are, respectively,

$$\operatorname{tr}\tilde{\mathscr{X}}^{w}.\boldsymbol{J}(\boldsymbol{X}) = \operatorname{tr}\boldsymbol{X}^{w}.\boldsymbol{J},$$

$$\operatorname{tr}\tilde{\boldsymbol{L}}^{w^*}.\boldsymbol{J}(\boldsymbol{X}) = \operatorname{tr}(\boldsymbol{I} - \boldsymbol{X}^{w})^{-1}.\boldsymbol{J}.$$

4.3. Necklaces and multiset permutations

The unlabeled structures corresponding to Eulerian cycles are the necklaces. Similarly, by unlabeling an Eulerian permutation we obtain a multiset permutation.

By Eqs. (51) and (52), we obtain the generating functions of the necklaces and multiset permutations, respectively.

Theorem 4.1.

$$\tilde{\mathscr{C}}(\boldsymbol{X}) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln\left|\frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}^{\underline{k}}}\right|, \tag{73}$$

$$\tilde{\mathscr{S}}(\boldsymbol{X}) = \prod_{k=1}^{\infty} \left|\frac{\boldsymbol{I}}{\boldsymbol{I} - \boldsymbol{X}^{\underline{k}}}\right|. \tag{74}$$

An easy consequence of the previous theorem is the following proposition

Proposition 4.1. (1) *The number of necklaces on $[n]$ with m_i occurrences of $i \in [n]$ and with l rises is*

$$[r^l \boldsymbol{x}^{\boldsymbol{m}}] \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln\left(1 - \frac{\prod_{i=1}^{n}\{1 + (r^k - 1)x_i^k\} - 1}{r^k - 1}\right)^{-1}. \tag{75}$$

(2) *The number of multiset permutations on $[n]$ with m_i occurrences of $i \in [n]$ and with l rises is*

$$[r^l.\boldsymbol{x}^{\boldsymbol{m}}] \prod_{k=1}^{\infty} \left(1 - \frac{\prod_{i=1}^{n}\{1 + (r^k - 1)x_i^k\} - 1}{r^k - 1}\right)^{-1}. \tag{76}$$

Proof. All we have to see is that Eqs. (75) and (76) follow, respectively, from (73) and (74) after substituting $x_{i,j}$ by $x_i r$ if $i < j$ and by x_i otherwise. After this substitution the determinant $\left|\boldsymbol{I}-\boldsymbol{X}^{\underline{k}}\right|$ becomes

$$\left|\begin{array}{cccc} 1-x_1^k & -x_1^k r^k & \cdots & -x_1^k r^k \\ -x_2^k & 1-x_2^k & \cdots & -x_2^k r^k \\ \cdots & \cdots & \cdots & \cdots \\ -x_n^k & -x_n^k & \cdots & 1-x_n^k \end{array}\right|. \tag{77}$$

Expanding by the first column, and using induction we get that it is equal to

$$1-\frac{\prod_{i=1}^n \{1+(r^k-1)x_i^k\}-1}{r^k-1}.$$

The first part of the previous proposition was obtained by Goulden and Jackson [5, Theorem 6.1] using classical Pólya theory.

A necklace $(i_1,i_2,\ldots,i_k)$ on $[n]$ is said to have η-pattern $(w)=(C_{r_1}C_{r_2}\cdots C_{r_k})$, if $([(i_1,i_2)],[(i_2,i_3)],\ldots,[(i_k,i_1)])=(w)$. A multiset permutation is said to have η-pattern in (w^*) if each necklace in it has η pattern in (w^*). By Eqs. (57), and (58), if (w) is a primitive necklace, the generating functions of necklaces and multiset permutations with η-pattern in (w^*) are

$$\tilde{\mathscr{C}}^{(w^*)}(\boldsymbol{X})=\sum_{k=1}^{\infty}\frac{\phi(k)}{k}\ln\left|\frac{\boldsymbol{I}}{\boldsymbol{I}-(\boldsymbol{X}^{\underline{k}})^w}\right|, \tag{78}$$

$$\tilde{\mathscr{S}}^{(w^*)}(\boldsymbol{X})=\prod_{k=1}^{\infty}\left|\frac{\boldsymbol{I}}{\boldsymbol{I}-(\boldsymbol{X}^{\underline{k}})^w}\right|. \tag{79}$$

4.4. Rearrangements of words

The substitution of $\boldsymbol{L}_0$ in $\mathfrak{R}$ is the species of rearrangements of Eulerian paths $\mathfrak{R}(\boldsymbol{L}_0)$. The corresponding unlabeled structures are rearrangements of words over $\{1,2,\ldots,n\}$. A rearrangement of words is a list of non-empty words

$$w_1|w_2|\ldots|w_k,$$

where

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix}$$

is a rearrangement, a_i and b_i being, respectively, the first and the last symbol in w_i, $i=1,2,\ldots,k$.

The generating function of that kind of unlabeled structures is

$$\widetilde{\mathfrak{R}(\boldsymbol{L}_0)}(\boldsymbol{X})=\left|\frac{\boldsymbol{I}-\boldsymbol{X}}{\boldsymbol{I}-2\boldsymbol{X}}\right|. \tag{80}$$

4.5. Necklaces of words, and multisets of necklaces of words

The unlabeled structure corresponding to an octopus is a necklace of words. For example, to the octopus in Fig. 8 corresponds the necklace of words

$$(121|123|332|21) = (123|332|21|121) = \cdots .$$

The unlabeled structure corresponding to an assembly of octopuses is a multiset of necklaces of words. We obtain from Eqs. (64) and (65)

$$\tilde{\mathrm{Oc}}(\boldsymbol{X}) = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \ln \left| \frac{\boldsymbol{I} - \boldsymbol{X}^{\underline{k}}}{\boldsymbol{I} - 2\boldsymbol{X}^{\underline{k}}} \right|, \tag{81}$$

$$\widetilde{U(\mathrm{Oc})}(\boldsymbol{X}) = \prod_{k=1}^{\infty} \left| \frac{\boldsymbol{I} - \boldsymbol{X}^{\underline{k}}}{\boldsymbol{I} - 2\boldsymbol{X}^{\underline{k}}} \right|. \tag{82}$$

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